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Many-Times Formalism and Coulomb Interaction

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Abstract

We extend here the many-times formalism, formerly used mainly for particles moving in given classical fields, to interacting particles. In order to minimize the difficulties associated with an equal-time interaction, we limit ourselves to nonrelativistic quantum mechanics and a two-particle interaction, such as that corresponding to the Coulomb force between charged particles. We obtain a set of differential equations which are really not consistent, but they serve as a guide to a formulation in terms of integral equations that has the same perturbation expansion as the usual theory for the scattering of particles. The integral equation for two-particle amplitudes can be modified to give the correct theory for bound states, but this is not the case for more than two particles. We expect that this theory can be generalized to a formulation of relativistic quantum mechanics of interacting particles.

1. *Introduction*

The generalization of nonrelativistic quantum mechanics of several particles to a relativistic theory presented the problem of expressing Lorentz covariance of functions of several three-vector variables and a single time. The use of functions of four-vector variables associates a different time parameter to each particle, giving rise to the many-times formalism (Dirac, 1932; Dirac *et al.*, 1932; Bloch, 1934; Tomonaga, 1946). This theory applied to the interaction of electrons, described by the Dirac equation or a nonrelativistic approximation of it, with the electromagnetic field. There were many difficulties in this formulation, and it was essentially abandoned in this context due to the problem of describing pair creation and annihilation, which seemed to require a variable number of time parameters. Relativistic quantum mechanics was relegated to the background in favor of the quantum theory of fields, which was surprisingly successful in quantum electrodynamics in spite of the mathematically ill-defined renormalization program for divergent diagrams.

Recently (Marx, 1969, 1790a, b) we showed how the wave functions of relativistic quantum mechanics can be interpreted in such a way that pair creation and annihilation are taken into account by changes in the mode of propagation of the 'particles' (Stueckelberg, 1941, 1942; Feynman, 1949). A 'particle' propagating forward in time is observed as a particle; propagating backward in time it corresponds to the antiparticle. The Klein–Gordon equation, which leads to a conserved charge that is the difference between two positive terms, is well suited for such a theory. On the other hand, the Dirac equation has to be modified to avoid problms with a positive 'charge'. In analogy to the nonrelativistic theory, a many-particle formalism can be expressed in terms of quantized field operators (Marx, 1972a), but the different modes of propagation of particles and antiparticles require a state vector dependent on several time parameters. The use of the many-times formalism in quantum field theory requires changes in the nature of the Fock space and the corresponding creation and annihilation operators (Marx, 1972c).

These applications of the many-times formalism were all limited to noninteracting charged particles in an external electromagnetic field, that is, we neglected the Coulomb interaction between particles and the radiation of electromagnetic waves. Under these restrictions, the Schrödinger equations of motion have the form

$$i\partial\Psi/\partial t_1 = H_1\Psi \tag{1.1}$$

$$i\partial\Psi/\partial t_2 = H_2\Psi \tag{1.2}$$

for two particles. The commutativity of the second derivatives with respect to time leads to the consistency condition

$$[H_1, H_2]\Psi = 0 \tag{1.3}$$

which is trivially satisfied when H_i depends only on the four-vector $x_{i\mu}$. The integrability conditions in a more general case are discussed by Bloch (1934); he presents the case for an interaction mediated by a dynamical electromagnetic field, and the conditions reduce to restrictions on the separations of localized particles.

In our approach to the interactions of charged particles and the electromagnetic field, we separate the Coulomb interaction and the radiation effects in a gauge-independent, observer-dependent manner (Goldberg & Marx, 1968; Marx, 1970c, 1972d). In this manner, the components of a state vector are functions of the particle variables $\mathbf{x}_i$ or $\mathbf{k}_i$ and functionals of the transverse components of the vector potential, with a time parameter associated to each particle and the field. The interactions are present only for equal times, and equations of the form (1.1) and (1.2) are difficult to write down, while consistency conditions such as (1.3) become essentially intractable. A somewhat different approach (Bethe & Salpeter, 1951, 1957) to a relativistic two-particle interaction is based on a wave equation of the form

$$M_1 M_2 \Psi = iG\Psi \tag{1.4}$$

where M_1 and M_2 are operators and G represents the interaction. This is closer to the formulation we use here.

In order to gain insight into such a general many-times formalism, we reformulate here the restricted problem of the Coulomb interaction of identical charged particles in a nonrelativistic approximation. Although the problem does not have great intrinsic importance, it has the advantage of a well-established single-time theory for comparison of the results.

The mathematical basis of our calculations is limited as a rule to formal computations, as is customary in this context. There are special problems associated to products of distributions (Schwartz, 1954), as can also be found in the usual formulation of quantum electrodynamics related to products of Green functions (Schwinger, 1959; Goldberg & Marx, 1967).

We begin in Section 2 with a brief analysis of the classical theory, where it is simple to introduce a different time variable for each particle in the Hamiltonian formulation. We briefly present the Green functions we need in Section 3, both for the time-dependent and time-independent problems. In Section 4 we recall to a limited extent the usual single-time theory; we give the perturbation expansion for a time-dependent scattering problem and the transformation to an integral equation of the bound-state problem. We discuss the general problem of the formulation of the many-times theory in terms of differential equations in Section 5 and, although we do not find a satisfactory answer for the general case, we obtain an equation that can be formally used in connection with the corresponding Green functions to obtain a perturbation expansion. We compare the terms in the perturbation expansions for both theories in the cases of two, three and four particles in Sections 6, 7 and 8 respectively, and we find that, to get complete agreement for more than two particles, the integral equations have to be modified further. In Section 9 we present the problem of bound states, which is particularly troublesome in many-times theories and relativistic quantum mechanics; we find a well-defined answer for two particles, but there is no simple way to generalize it to three or more particles. We present concluding remarks in Section 10.

We use natural units, that is, we set $\hbar = 1$ and $\varepsilon_0 = 1$.

To keep the complication of the equations to a minimum, we assume that we have identical particles of mass m and charge e. This is not a fundamental limitation and the generalization to different types of particles is straightforward.

In order to shorten the equations and facilitate the work in the more complicated cases, we introduce abbreviations for the arguments of functions, when convenient; we replace

$$\mathbf{x}_1 \to \mathbf{1}, \qquad t_1 \to 1, \qquad \mathbf{x}_2' \to \mathbf{2}', \qquad t_f \to f \qquad d^3x_1 \to d\mathbf{1}, \qquad dt_1 \to d1 \tag{1.5}$$

and so on.

2. *Classical Theory*

Although the concepts of space-time and world lines for classical particles came into use through the theory of relativity, the basic ideas apply equally well to a nonrelativistic problem. Similarly, there is no conceptual difficulty in assigning a different time parameter to each particle in a case such as the nonrelativistic Coulomb interaction.

We do not apply directly the results of this classical problem to its generalization in quantum mechanics, but we find it useful to indicate the new lines of thought introduced by the many-times formalism, especially in Hamiltonian dynamics.

The usual theory of n interacting charged particles can be obtained from the Lagrangian

$$L=\sum_{j=1}^{n}\frac{m\dot{\mathbf{x}}_j^2}{2}-\sum_{j=2}^{n}\sum_{j'=1}^{j-1}\frac{e^2}{4\pi|\mathbf{x}_j-\mathbf{x}_{j'}|} \tag{2.1}$$

where all $\mathbf{x}_j$ are functions of a single variable t. The Hamiltonian derived from this Lagrangian is

$$H=\sum_{j=1}^{n}\frac{\mathbf{p}_j^2}{2m}+\sum_{j=2}^{n}\sum_{j'=1}^{j-1}\frac{e^2}{4\pi|\mathbf{x}_j-\mathbf{x}_{j'}|} \tag{2.2}$$

and the corresponding equations of motion are

$$\dot{\mathbf{x}}_j=\mathbf{p}_j/m \tag{2.3}$$

$$\dot{\mathbf{p}}_j=\sum_{j'=1}^{n}{}'\frac{e^2(\mathbf{x}_j-\mathbf{x}_{j'})}{4\pi|\mathbf{x}_j-\mathbf{x}_{j'}|^3} \tag{2.4}$$

where the prime on the summation symbol indicates that we exclude $j'=j$.

Alternatively, it is possible to consider the position of each particle as a function of a different time variable. The equations of motion, rewritten in the form

$$\frac{d^2\mathbf{x}_j(t_j)}{dt_j^2}=\sum_{j'=1}^{n}{}'\int_{t_j}^{\infty}\frac{e^2[\mathbf{x}_j(t_j)-\mathbf{x}_{j'}(t_{j'})]}{4\pi|\mathbf{x}_j(t_j)-\mathbf{x}_{j'}(t_{j'})|^3}\delta(t_j-t_{j'})\,dt_{j'} \tag{2.5}$$

can be obtained from a set of single-particle Hamiltonians

$$H_j=\frac{\mathbf{p}_j^2}{2m}+\sum_{j'=1}^{n}{}'\frac{e^2}{4\pi|\mathbf{x}_j-\mathbf{x}_{j'}|} \tag{2.6}$$

and we note that

$$\sum H_j\neq H \tag{2.7}$$

The more general classical problem can be formulated in terms of an action of the form (Rzewuski, 1964; Rohrlich, 1965)

$$I=\sum_{j=1}^{n}\int L_{jj}\,d\tau_j+\tfrac{1}{2}\sum_{j=1}^{n}\sum_{j'=1}^{n}{}'\int L_{jj'}\,d\tau_j\,d\tau_{j'} \tag{2.8}$$

where $\mathbf{x}_j$ and t_j are functions of a parameter τ_j, the L_{jj} are functionals of $\mathbf{x}_j$, t_j, $d\mathbf{x}_j/d\tau_j$ and $dt_j/d\tau_j$ and the $L_{jj'}$ are functionals of these variables with indices j or j'. In a relativistically covariant formulation, one difficulty is the specification of the limits of integration if they are finite. The equations of motion are the Euler–Lagrange equations for the Lagrangians

$$L_j = L_{jj} + \sum_{j'=1}^{n}{}' \int L_{jj'}\, d\tau_{j'} \tag{2.9}$$

and further difficulties arise in the Hamiltonian formulation due to the definition of momenta when $L_{jj'}$ depends on the velocities, and to the presence of constraints coming from the invariance under parameter changes. If the $L_{jj'}$ depend on velocities, the definition

$$H_j = \mathbf{p}_j . \dot{\mathbf{x}}_j - L_j \tag{2.10}$$

for $\tau_j = t_j$ does not assure elimination of the velocities from H_j. In the present case, it leads correctly to equation (2.6).

3. *Green Functions*

We show how the Green functions of interest to our problem are obtained and derive some useful relationships.

The Green function for the time-dependent Schrödinger equation for n particles has to satisfy

$$\left[i\,\partial/\partial t + (1/2m)\sum_{j=1}^{n} \nabla_j^2\right] G(\mathbf{1},\ldots,\mathbf{n},t;\mathbf{1}',\ldots,\mathbf{n}',t') = -\delta(\mathbf{1}-\mathbf{1}')\ldots\delta(\mathbf{n}-\mathbf{n}')\,\delta(t-t') \tag{3.1}$$

This Green function depends only on the differences $\mathbf{x}_j - \mathbf{x}_j'$ and $t - t'$, so that we only have to determine it for $\mathbf{x}_j' = 0$, $t' = 0$. The solution is obtained from the Fourier transform, and is

$$G_R(\mathbf{1},\ldots,\mathbf{n};t) = -\frac{1}{(2\pi)^{3n+1}} \int \frac{d^3k_1\ldots d^3k_n\, d\omega \exp\left[i(\sum \mathbf{k}_j . \mathbf{x}_j - \omega t)\right]}{\omega - \sum \mathbf{k}_j^2/(2m + i\varepsilon)} \tag{3.2}$$

where the (small) positive quantity ε specifies how the contour has to be deformed around the pole in the complex ω plane in order to obtain the retarded Green function. It vanishes for $t < 0$ and, for $t > 0$,

$$G_R(\mathbf{1},\ldots,\mathbf{n};t) = \frac{i}{(2\pi)^{3n}} \int d^3k_1\ldots d^3k_n \exp\left[i \sum_{j=1}^{n} \left(\mathbf{k}_j . \mathbf{x}_j - \frac{\mathbf{k}_j^2 t}{2m + i\varepsilon}\right)\right] \tag{3.3}$$

We do the angular integrations to derive

$$\int d^3k \exp\left[i\left(\mathbf{k}.\mathbf{x} - \frac{\mathbf{k}^2 t}{2m + i\varepsilon}\right)\right] = \frac{4\pi}{r} \int_0^\infty k\, dk \sin(kr) \exp\left(-i\frac{k^2 t}{2m + i\varepsilon}\right) \tag{3.4}$$

and use equation (2.4.19) in Erdélyi *et al.* (1954) to show that

$$\int d^3k \exp\left[i\left(\mathbf{k}.\mathbf{x} - \frac{\mathbf{k}^2 t}{2m + i\varepsilon}\right)\right] = \left(\frac{2\pi m}{it}\right)^{3/2} \exp\left(i\frac{m\mathbf{x}^2}{2t}\right) \tag{3.5}$$

where we have set $\varepsilon = 0$ after using it to satisfy the requirement on the argument of the coefficient of ik^2. Consequently,

$$G_R(\mathbf{1}, \ldots, \mathbf{n}; t) = i\left(\frac{m}{2\pi i t}\right)^{(3n)/2} \exp\left(i\frac{m}{2t}\sum_{j=1}^{n} \mathbf{x}_j{}^2\right)\theta(t) \tag{3.6}$$

where

$$\theta(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geqslant 0 \end{cases} \tag{3.7}$$

For a single particle, it reduces to

$$G_R(\mathbf{x}, t) = i\left(\frac{m}{2\pi i t}\right)^{3/2} \exp\left(\frac{im\mathbf{x}^2}{2t}\right)\theta(t) \tag{3.8}$$

whence

$$G_R(\mathbf{1}, \ldots, \mathbf{n}; t) = i^{-n+1} \prod_{j=1}^{n} G_R(\mathbf{x}_j, t) \tag{3.9}$$

This equation is valid if

$$[\theta(t)]^n = \theta(t) \tag{3.10}$$

which follows from equation (3.7) where we chose $\theta(0) = 1$. On the other hand, we also have to use

$$d\theta(t)/dt = \delta(t) \tag{3.11}$$

which is really a relationship between distributions, and in this context equation (3.10) is ill-defined and might lead to an extraneous factor n when the differential operator is applied to both sides of equation (3.9).

If we set $t = 0$ in equation (3.3), we find

$$G_R(\mathbf{1}, \ldots, \mathbf{n}; 0) = i\delta(\mathbf{1}) \ldots \delta(\mathbf{n}) \tag{3.12}$$

A direct calculation using equation (3.5) shows that

$$\int d^3x' \, G_R(\mathbf{x} - \mathbf{x}', t - t') G_R(\mathbf{x}' - \mathbf{x}'', t' - t'') = i\theta(t - t')\theta(t' - t'') G_R(\mathbf{x} - \mathbf{x}'', t - t'') \tag{3.13}$$

Another Green function of interest is the one for the time-independent Schrödinger equation, which depends on a parameter E and satisfies

$$[E + (1/2m)\sum \nabla_j{}^2] G(\mathbf{1}, \ldots, \mathbf{n}; E) = -\delta(\mathbf{1}) \ldots \delta(\mathbf{n}) \tag{3.14}$$

whence

$$G(\mathbf{1}, \ldots, \mathbf{n}; E) = \frac{1}{(2\pi)^{3n}} \int \frac{d^3k_1 \ldots d^3k_n \exp(i\sum \mathbf{k}_j.\mathbf{x}_j)}{\sum \mathbf{k}_j{}^2/(2m) - E} \tag{3.15}$$

For bound states, we have $E < 0$ and there are no problems with poles of the integrand when doing the integrations. After doing the angular integrations, we have

$$G(\mathbf{1},\ldots,\mathbf{n};E) = \frac{1}{(2\pi^2)^n r_1 \ldots r_n} \int_0^\infty \frac{k_1\,dk_1 \ldots k_n\,dk_n \sin(k_1 r_1)\ldots\sin(k_n r_n)}{\sum k_j^2/(2m) - E} \tag{3.16}$$

and we integrate this by using equations (2.2.15), (2.4.35) and (2.13.45) in Erdélyi *et al.* (1954). We find

$$G(\mathbf{1};E) = (m/2\pi r_1)\exp[-(-2mE)^{1/2} r_1] \tag{3.17}$$

$$G(\mathbf{1},\mathbf{2};E) = \frac{m^2(-E)}{2\pi^3(\mathbf{x}_1^2 + \mathbf{x}_2^2)} K_2[(-2mE)^{1/2}(\mathbf{x}_1^2 + \mathbf{x}_2^2)^{1/2}] \tag{3.18}$$

and, for $n > 2$,

$$G(\mathbf{1},\ldots,\mathbf{n};E) = \frac{2m(-2mE)^{(3n/4)-1/2}}{(2\pi)^{(3n)/2}(\sum \mathbf{x}_j^2)^{(3n/4)-1/2}} K_{-(3n/2)+1}[(-2mE)^{1/2}(\sum \mathbf{x}_j^2)^{1/2}] \tag{3.19}$$

where the K_ν are modified Bessel functions.

4. *Single-Time Theory*

The time-dependent Schrödinger equation for n particles is

$$M\Psi = \alpha H_I \Psi \tag{4.1}$$

where

$$M \equiv i\partial/\partial t + (1/2m)\sum \nabla_j^2 \tag{4.2}$$

$$H_I = \sum_{j=2}^{n} \sum_{j'=1}^{j-1} |\mathbf{x}_j - \mathbf{x}_{j'}|^{-1} \tag{4.3}$$

$$\alpha = e^2/4\pi \tag{4.4}$$

In a scattering problem, we assume that Ψ is given at the initial time t_i and determine it for later times. To obtain a perturbation expansion in powers of α, we first find the corresponding integral equation. We assume that all functions vanish at spatial infinity sufficiently rapidly so that no contributions from integrals over surfaces at infinity have to be taken into account. The appropriate form of Green's theorem is

$$\int_i^f dt \int d\mathbf{1}\ldots d\mathbf{n}\Phi\overleftrightarrow{M}\Psi = i\int d\mathbf{1}\ldots d\mathbf{n}[\Phi\Psi]_i^f \tag{4.5}$$

where

$$\Phi\overleftrightarrow{M}\Psi \equiv \Phi M\Psi - \Psi M^*\Phi \tag{4.6}$$

We now use primed variables in equation (4.5) and set

$$\Phi(\mathbf{1}',\ldots,\mathbf{n}';t') = G_R(\mathbf{1}-\mathbf{1}',\ldots,\mathbf{n}-\mathbf{n}';t-t') \tag{4.7}$$

to find the integral equation, for $t < t_f$,

$$\Psi(1,\ldots,n;t) = -\alpha \int_{t_i}^{t} dt' \int d1' \ldots dn'\, G_R(1-1',\ldots,n-n';t-t') \times H_I(1',\ldots,n')\,\Psi(1',\ldots,n';t') - i\int d1'\ldots dn' \times G_R(1-1',\ldots,n-n';t-t_i)\,\Psi_i(1',\ldots,n') \tag{4.8}$$

where Ψ_i is the given initial value for Ψ. We now set

$$\Psi = \sum_{p=0}^{\infty} \alpha^p \Psi^{(p)} \tag{4.9}$$

Then $\Psi^{(0)}$ is obtained from the last term only in equation (4.8), that is,

$$\Psi^{(0)}(1,\ldots,n;t) = -i\int d1'\ldots dn'\, G_R(1-1',\ldots,n-n';t-t')\,\Psi_i(1',\ldots,n') \tag{4.10}$$

while higher-order terms satisfy homogeneous initial conditions and are obtained from the recursion relation, for $p \geqslant 1$,

$$\Psi^{(p)}(1,\ldots,n;t) = -\int_{t_i}^{t} dt' \int d1'\,..dn'\, G_R(1-1',\ldots,n-n';t-t') \times H_I(1',\ldots,n')\,\Psi^{(p-1)}(1',\ldots,n';t') \tag{4.11}$$

It is clear from these expressions that Ψ will have the same symmetry as Ψ_i.

The problem of the bound state leads to a different type of integral equation. We separate the time dependence of the wave function and write

$$\Psi(1,\ldots,n;t) = \psi(1,\ldots,n)\exp(-iEt) \tag{4.12}$$

The Schrödinger equation reduces to

$$(H_0 - E)\psi = -\alpha H_I \psi, \qquad E < 0 \tag{4.13}$$

and we use the corresponding Green function to obtain

$$\psi(1,\ldots,n) = -\alpha \int d1'\ldots dn'\, G(1-1',\ldots,n-n';E) \times H_I(1',\ldots,n')\,\psi(1',\ldots,n') \tag{4.14}$$

This is a homogeneous equation and the problem reduces to the determination of those values of E that allow for a nontrivial solution for ψ.

This equation can also be obtained from the time-dependent approach. We substitute equation (4.12) into (4.8) and let $t_i \to -\infty$. The Green function (3.6) tends to zero, and we obtain

$$\psi(1,\ldots,n) = -\alpha \int_{-\infty}^{t} dt' \int d1'\ldots dn'\, G_R(1-1',\ldots,n-n';t-t') \times H_I(1',\ldots,n')\,\Psi(1',\ldots,n')\exp[iE(t-t')] \tag{4.15}$$

We perform the time integration by setting

$$t - t' = u \tag{4.16}$$

and find

$$\int_0^\infty du G_R(\mathbf{1}, \ldots, \mathbf{n}; u) \exp(iEu)$$
$$= -\int_{-\infty}^{\infty} du \frac{\exp(iEu)}{(2\pi)^{3n+1}} \int \frac{d^3k_1 \ldots d^3k_n d\omega \exp[i(\sum \mathbf{k}_j . \mathbf{x}_j - \omega u)]}{\omega - \sum \mathbf{k}_j^2/(2m + i\varepsilon)} \tag{4.17}$$

since $G_R(u) = 0$ for $u < 0$. We interchange orders of integration and use the Dirac δ-function to show that

$$\int_0^\infty du\, G_R(\mathbf{1}, \ldots, \mathbf{n}; u) \exp(iEu) = \frac{1}{(2\pi)^{3n}} \int \frac{d^3k_1 \ldots d^3k_n \exp(i \sum \mathbf{k}_j . \mathbf{x}_j)}{\sum \mathbf{k}_j^2/(2m + i\varepsilon) - E} \tag{4.18}$$

and, comparing with equation (3.15), we see that the integral equation (4.15) reduces to the time-independent one (4.14).

5. *Many-Times Formalism*

We first examine the possibility of using equations of the form

$$i\partial \Psi(\mathbf{1}1, \ldots, \mathbf{n}n)/\partial t_j = H_j \Psi(\mathbf{1}1, \ldots, \mathbf{n}n) \tag{5.1}$$

where H_j can be obtained, for instance, from equation (2.6) in the classical theory. The first objection to such an approach is that

$$[H_j, H_{j'}] \neq 0, \qquad j \neq j' \tag{5.2}$$

which would impose serious restrictions on the wave function through equation (1.3). We could still rewrite equation (5.1) in the form

$$M_j \Psi = \alpha H_{1j} \Psi \tag{5.3}$$

where

$$M_j \equiv i\partial/\partial t_j + \nabla_j^2/(2m) \tag{5.4}$$

and use it to obtain an integral equation. Since the consistency conditions are not satisfied, the solution would not obey the differential equations (5.1) that were initially postulated. To explore this possibility further, we study the two-particle amplitude and use

$$\int_i^f d\mathbf{1}\, d\mathbf{2} \int d\mathbf{1}\, d\mathbf{2} \Phi \overleftrightarrow{M}_1 \overleftrightarrow{M}_2 \Psi = i \int d\mathbf{1}\, d\mathbf{2} [\Phi \Psi]_i^f \tag{5.5}$$

$$\Phi(\mathbf{1}'\,1', \mathbf{2}'\,2') = G_R(\mathbf{1} - \mathbf{1}', 1 - 1') G_R(\mathbf{2} - \mathbf{2}', 2 - 2') \tag{5.6}$$

to obtain

$$\Psi(11,22) = -\alpha \int_i^f d1' \int d\mathbf{1}'\, G_R(\mathbf{1}-\mathbf{1}', 1-1')\, H_I(\mathbf{1}',2)\, \Psi(\mathbf{1}'\,1', 22)$$
$$- \alpha \int_i^f d2' \int d\mathbf{2}'\, G_R(\mathbf{2}-\mathbf{2}', 2-2')\, H_I(\mathbf{1},\mathbf{2}')\, \Psi(\mathbf{11}, \mathbf{2}'\,2')$$
$$- \alpha^2 \int_i^f d1'\, d2' \int d\mathbf{1}'\, d\mathbf{2}'\, G_R(\mathbf{1}-\mathbf{1}', 1-1')\, G_R(\mathbf{2}-\mathbf{2}', 2-2')$$
$$\times [H_I(\mathbf{1}',\mathbf{2}')]^2\, \Psi(\mathbf{1}'\,1', \mathbf{2}'\,2') - \int d\mathbf{1}'\, d\mathbf{2}'\, G_R(\mathbf{1}-\mathbf{1}', 1-i)$$
$$\times G_R(\mathbf{2}-\mathbf{2}', 2-i)\, \Psi_i(\mathbf{1}', \mathbf{2}') \tag{5.7}$$

From this equation we find

$$\Psi^{(0)}(\mathbf{11},\mathbf{22}) = -\int d\mathbf{1}'\, d\mathbf{2}'\, G_R(\mathbf{1}-\mathbf{1}', 1-i)\, G_R(\mathbf{2}-\mathbf{2}', 2-i)\, \Psi_i(\mathbf{1}', \mathbf{2}'), \tag{5.8}$$

$$\Psi^{(1)}(\mathbf{11},\mathbf{22}) = -\int_i^1 d1' \int d\mathbf{1}'\, G_R(\mathbf{1}-\mathbf{1}', 1-1')\, H_I(\mathbf{1}',2)\, \Psi^{(0)}(\mathbf{1}'\,1', \mathbf{22})$$
$$- \int_i^2 d2' \int d\mathbf{2}'\, G_R(\mathbf{2}-\mathbf{2}', 2-2')\, H_I(\mathbf{1},\mathbf{2}')\, \Psi^{(0)}(\mathbf{11}, \mathbf{2}'\,2') \tag{5.9}$$

and so on. It is easy to verify that the equal-time amplitudes obtained by setting $t_1 = t_2 = t$ agree with those obtained from equations (4.10) and (4.11) only for $\Psi^{(0)}$. This is not surprising if we consider that the Coulomb interaction is effective only when the times are equal, which is not the case with equation (5.1).

Actually, the equal-time nature of the interaction makes it singular when used in a many-times approach. We cannot simply multiply the terms in H_{IJ} by a set of Dirac δ-functions $\delta(t_j - t_{j'})$ since, among other problems, the equation would be dimensionally inconsistent; also it does not help to set the times equal to each other only in the right-hand side.

We thus prefer to abandon equations of the form (5.1)† and try those like the Bethe–Salpeter equation (1.4). We somehow have to generalize again‡ the concept of the Hamiltonian as a time-displacement operator. We have to go to second-order terms and write

$$\Psi(t_1 + dt_1, \ldots, t_n + dt_n) - \Psi(t_1, \ldots, t_n)$$
$$\approx \left[-\sum_{j=1}^{n} iH_{0j}\, dt_j + \tfrac{1}{2}\left(-\sum_{j=1}^{n} iH_{0j}\, dt_j \right)^2 - \alpha \sum_{j=2}^{n} \sum_{j'=1}^{j-1} H_{IJ'}\, \delta(t_j - t_{j'})\, dt_j\, dt_{j'} \right]$$
$$\times \Psi(t_1, \ldots, t_n) \tag{5.10}$$

† Further developments of a relativistic quantum mechanics for two interacting particles can be found in Fronsdal & Lundberg (1970) and Fronsdal (1971).

‡ We already had to introduce a generalization for the relativistic problem (Marx, 1972a).

where

$$H_{jj'} = |\mathbf{x}_j - \mathbf{x}_{j'}|^{-1} \tag{5.11}$$

The interaction term is highly singular, and we are not prepared to deal in a mathematically consistent way with such an expression. We thus limit ourselves to formal expansions. In this we are guided by our experience with the many-times formalism for noninteracting particles and the single-time approach of Section 4.

If we allow only one of the dt_j to be different from zero, we obtain

$$M_j \Psi = 0 \tag{5.12}$$

which corresponds to free particles and should be valid unless the times are equal. If we now have two of the dt_j different from zero, we obtain the second-order equation of the form (1.4),

$$(M_j M_{j'}) \Psi = iH_{jj'} \delta(t_j - t_{j'}) \Psi, \qquad j \neq j' \tag{5.13}$$

which is compatible with equation (5.12) only for different times. Furthermore, if we have more than two particles, we still have problems with consistency conditions, since

$$M_1 M_3 (M_1 M_2 \Psi) = iM_1 H_{12}(1,2) \delta(1-2) M_3 \Psi \tag{5.14}$$

is not the same as

$$M_1 M_2 (M_1 M_3 \Psi) = iM_1 H_{13}(1,3) \delta(1-3) M_2 \Psi \tag{5.15}$$

These observations, as well as other problems in quantum electrodynamics, suggest that difficulties with products of distributions might be traced to a lack of associativity.

We now use equation (5.13) as a guideline to find integral equations for two-, three- and four-particle amplitudes. The problems with the consistency equations will require some further changes, and the resulting amplitudes do not satisfy equation (5.12) and (5.13). On the other hand, we show that the perturbation expansions agree with those in Section 4. We do not present the general case because the notation becomes too unwieldy, but the procedures can easily be generalized.

6. *Two-Particle Amplitudes*

Since equation (5.12) does not give an interaction, we cannot use Green's theorem in the form of equation (5.5). Instead, we use

$$\int d1\, d2 \Phi \overleftrightarrow{M_1 M_2} \Psi$$

$$= \int d1\, d2 [i(\partial/\partial t_2)(\Psi M_1^* \Phi) + i(\partial/\partial t_1)(\Psi M_2^* \Phi) - (\partial^2/\partial t_1\, \partial t_2)(\Phi\Psi)] \tag{6.1}$$

where

$$\Phi \overleftrightarrow{M_1 M_2} \Psi \equiv \Phi M_1 M_2 \Psi - \Psi M_1^* M_2^* \Phi \tag{6.2}$$

We integrate both over t_1 and t_2 from t_i to t_f to obtain

$$\int_i^f d1\, d2 \int d1\, d2 \Phi \overleftrightarrow{M_1 M_2} \Psi = i \int_i^f d1 \int d1\, d2 [\Psi M_1^* \Phi]_{2=i}^{2=f} + i \int_i^f d2 \int d1 \times d2 [\Psi M_2^* \Phi]_{1=i}^{1=f} - \int d1\, d2 [\Phi \Psi]_{1,2=i}^{1,2=f} \tag{6.3}$$

We change over to primed variables of integration, set

$$\Phi(1'1', 2'2') = G_R(1-1', 1-1') G_R(2-2', 2-2') \tag{6.4}$$

in equation (6.3) and use equation (5.13) to derive the integral equation

$$\begin{aligned} \Psi(11, 22) = i\alpha \int_i^f dt' \int d1'\, d2'\, G_R(1-1', t_1-t') G_R(2-2', t_2-t') \\ \times H_{I12}(1', 2') \Psi(1't', 2't') - i \int d1'\, G_R(1-1', 1-i) \\ \times \Psi(1'i, 22) - i \int d2'\, G_R(2-2', 2-i) \Psi(11, 2'i) \\ + \int d1'\, d2'\, G_R(1-1', 1-i) G_R(2-2', 2-i) \Psi_i(1', 2') \end{aligned} \tag{6.5}$$

The time integration really extends only to the smaller of t_1 and t_2, due to the factor $\theta(t_1 - t') \times \theta(t_2 - t')$ from the retarded Green functions.

If we apply M_1 to both sides of equation (6.5), we get an integral equation for $M_1 \Psi$,

$$\begin{aligned} M_1 \Psi(11, 22) = i \int d2' [G_R(2-2', 2-1) \alpha H_{I12}(1, 2') \\ \times \Psi(11, 2'1) - G_R(2-2', 2-i) M_1 \Psi(11, 2'i)] \end{aligned} \tag{6.6}$$

If we let $t_2 \to t_i$, we obtain an identity; more generally, the first term of the integrand vanishes for $t_2 < t_1$. By applying the operator M_2 to both sides, we verify that equation (5.10) is indeed satisfied.

To find the terms in the perturbation expansion, we note that $\Psi^{(0)}$ has to satisfy

$$\begin{aligned} \Psi^{(0)}(11, 22) = -i \int d1'\, G_R(1-1', 1-i) \Psi^{(0)}(1'i, 22) - i \int d2' \\ \times G_R(2-2', 2-i) \Psi^{(0)}(11, 2'i) + \int d1'\, d2' \\ \times G_R(1-1', 1-i) G_R(2-2', 2-i) \Psi_i(1', 2') \end{aligned} \tag{6.7}$$

and the solution is given by equation (5.8). The higher-order contributions satisfy homogeneous initial conditions, and are given by

$$\Psi^{(p)}(11,22) = i\int_i^f dt' \int d1'\,d2'\,G_R(1-1',t_1-t') \times G_R(2-2',t_2-t')\,H_{I12}(1',2')\,\Psi^{(p-1)}(1't',2't') \quad (6.8)$$

They reduce to those given by equation (4.11) when we set $t_1 = t_2 = t$, $n = 2$ and use equation (3.9).

The lowest-order wave function obeys the homogeneous equations

$$M_1\,\Psi^{(0)} = M_2\,\Psi^{(0)} = 0 \quad (6.9)$$

while equation (6.8) gives

$$M_1\,\Psi^{(p)}(11,22) = -i\int d2'\,G_R(2-2',2-1)\,H_{I12}(1,2')\,\Psi^{(p-1)}(11,2'1) \quad (6.10)$$

It vanishes when $t_2 < t_1$, and for $t_2 = t_1$ we find

$$M_1\,\Psi^{(p)}(11,22)|_{2=1} = H_{I12}(1,2)\,\Psi^{(p-1)}(11,21) \quad (6.11)$$

which contradicts equation (4.1). This is another indication of problems with derivatives of products of θ-functions.

7. *Three-Particle Amplitudes*

We extend Green's theorem (6.3) to the case of functions of three sets of variables,

$$\begin{aligned}
&\int_i^f d1\,d2\,d3\int d1\,d2\,d3[(M_1^*\,\Phi)\,M_2\,M_3\,\Psi + (M_2^*\,\Phi)\,M_3\,M_1\,\Psi \\
&\qquad + (M_3^*\,\Phi)\,M_1\,M_2\,\Psi - 3\Psi M_1^*\,M_2^*\,M_3^*\,\Phi] \\
&= \int d1\,d2\,d3\Big\{2i\int_i^f d1\,d2[\Phi M_1\,M_2\,\Psi]_{3=t}^{3=f} + 2i\int_i^f d2\,d3[\Phi M_2\,M_3\,\Psi]_{1=t}^{1=f} \\
&\quad + 2i\int_i^f d3\,d1[\Phi M_3\,M_1\,\Psi]_{2=t}^{2=f} + 3\int_i^f d1[\Psi M_1^*\,\Phi]_{2,3=t}^{2,3=f} \\
&\quad + 3\int_i^f d2[\Psi M_2^*\,\Phi]_{3,1=t}^{3,1=f} + 3\int_i^f d3[\Psi M_3^*\,\Phi]_{1,2=t}^{1,2=f} + 6i[\Phi\Psi]_{1,2,3=t}^{1,2,3=f}\Big\}
\end{aligned} \quad (7.1)$$

We set

$$\Phi(1'1',2',2'3'3') = G_R(1-1',1-1')\,G_R(2-2',2-2')\,G_R(3-3',3-3') \quad (7.2)$$

and use equation (5.13) to obtain the integral equation

$$\begin{aligned}\Psi(11,22,33) &= \tfrac{1}{3}\Big\{i\alpha\int_i^f dt'\int d1'\,d2'\,G_R(1-1',t_1-t')\,G_R(2-2',t_2-t')\,H_{I12}(1',2')\\ &\quad\times\big[\Psi(1't',2't',33)-2i\int d3'\,G_R(3-3',3-i)\,\Psi(1't',2't',3'i)\big]\\ &\quad+(2\text{ permutations})-3\int d1'\,d2'\,G_R(1-1',1-i)\,G_R(2-2',2-i)\\ &\quad\times\Psi(1'i,2'i,33)+(2\text{ permutations})-6i\int d1'\,d2'\,d3'\\ &\quad\times G_R(1-1',1-i)\,G_R(2-2',2-i)\,G_R(3-3',3-i)\,\Psi_i(1',2',3')\end{aligned} \tag{7.3}$$

where the terms that are not written out are obtained by cyclic permutations of the indices. We note that, if we set $t_3 = t_i$, we obtain

$$\begin{aligned}\Psi(11,22,3i) &= i\alpha\int_i^f dt'\int d1'\,d2'\,G_R(1-1',t_1-t')\,G_R(2-2',t_2-t')\\ &\quad\times H_{I12}(1',2')\,\Psi(1't',2't',3i)-i\int d1'\,G_R(1-1',1-i)\\ &\quad\times\Psi(1'i,22,3i)-i\int d2'\,G_R(2-2',2-i)\,\Psi(11,2'i,3i)\\ &\quad+\int d1'\,d2'\,G_R(1-1',1-i)\,G_R(2-2',2-i)\,\Psi_i(1',2',3)\end{aligned} \tag{7.4}$$

which is precisely the equation for two-particle amplitudes (6.5).

We first find $\Psi^{(0)}$, which is a solution of the homogeneous equation and satisfies the initial condition. It is

$$\begin{aligned}\Psi^{(0)}(11,22,33) &= i\int d1'\,d2'\,d3'\,G_R(1-1',1-i)\\ &\quad\times G_R(2-2',2-i)\,G_R(3-3',3-i)\,\Psi_i(1',2',3')\end{aligned} \tag{7.5}$$

which satisfies equation (7.3) to lowest order. Higher-order terms satisfy homogeneous initial conditions, and the recursion relation we obtain is

$$\begin{aligned}\Psi^{(p)}(11,22,33) &= \tfrac{1}{3}i\int_i^f dt'\int d1'\,d2'\,G_R(1-1',t_1-t')\,G_R(2-2',t_2-t')\\ &\quad\times H_{I12}(1',2')\big[\Psi^{(p-1)}(1't',2't',33)-2i\int d3'\\ &\quad\times G_R(3-3',3-i)\,\Psi^{(p-1)}(1't',2't',3'i)\big]\\ &\quad+(2\text{ permutations})\end{aligned} \tag{7.6}$$

in particular

$$\Psi^{(1)}(11,22,33) = i\int_i^f dt' \int d1'\,d2'\,G_R(1-1',t_1-t')\,G_R(2-2',t_2-t')$$
$$\times H_{I12}(1',2')\,\Psi^{(0)}(1't',2't',33) + (2\text{ permutations}) \quad (7.7)$$

$$\Psi^{(2)}(11,22,33) = -\int_i^f dt' \int d1'\,d2'\,G_R(1-1',t_1-t')\,G_R(2-2',t_2-t')$$
$$\times H_{I12}(1',2')\Big[\int_i^f dt'' \int d1''\,d2''\,G_R(1'-1'',t'-t'')$$
$$\times G_R(2'-2'',t'-t'')\,H_{I12}(1'',2'')\,\Psi^{(0)}(1''t'',2''t'',33)$$
$$+\tfrac{1}{3}\int_i^f dt'' \int d2''\,d3''\,G_R(2'-2'',t'-t'')$$
$$\times G_R(3-3'',t_3-t'')\,H_{I23}(2'',3'')\,\Psi^{(0)}(1't',2''t'',3''t'')$$
$$+\tfrac{1}{3}\int_i^f dt'' \int d3''\,d1''\,G_R(3-3'',t_3-t'')$$
$$\times G_R(1'-1'',t'-t'')\,H_{I31}(3'',1'')\,\Psi^{(0)}(1''t'',2't',3''t'')\Big]$$
$$+ (2\text{ permutations}) \quad (7.8)$$

where we use equation (7.5) to show that

$$\int d3'\,G_R(3-3',3-i)\,\Psi^{(0)}(1't',2't',3'i) = \Psi^{(0)}(1't',2't',33) \quad (7.9)$$

Equation (7.6) shows that $\Psi^{(p)}$, $p>0$, vanishes when any two times are set equal to t_i. Furthermore, we use it to compute

$$M_1 M_2\,\Psi^{(p)}(11,22,33) = iH_{I12}(1,2)\,\delta(1-2)$$
$$\times\Big[\tfrac{1}{3}\Psi^{(p-1)}(11,22,33) - \tfrac{2}{3}i\int d3'\,G_R(3-3',3-i)\,\Psi^{(p-1)}(11,22,3'i)\Big]$$
$$+\tfrac{1}{3}i\int d3'[G_R(3-3',3-2)\,H_{I23}(2,3')\,M_1\,\Psi^{(p-1)}(11,22,3'2)$$
$$+ G_R(3-3',3-1)\,H_{I31}(3',1)\,M_2\,\Psi^{(p-1)}(11,22,3'1)] \quad (7.10)$$

where only the first term has the expected factor $\delta(t_1 - t_2)$. For equal times, $\Psi^{(1)}$ in equation (7.7) agrees with that obtained from equation (4.11) for $n = 3$. On the other hand,

$$\begin{aligned}\Psi^{(2)}(1,2,3;t) = -\int_{t_i}^{t} dt' \int d1'\,d2'\,G_R(1-1',t-t')\,G_R(2-2',t-t') \\ \times H_{I12}(1',2')\Big[\int_{t_i}^{t'} dt'' \int d1''\,d2''\,G_R(1'-1'',t'-t'') \\ \times G_R(2'-2'',t'-t'')\,\Psi^{(0)}(1''t'',2''t'',3t) + \int_{t_i}^{t'} dt'' \int d2'' \\ \times d3''\,G_R(2'-2'',t'-t'')\,G_R(3-3'',t-t'') \\ \times H_{I23}(2'',3'')\,\Psi^{(0)}(1't',2''t'',3''t'') + \int_{t_i}^{t'} dt'' \int d3''\,d1'' \\ \times G_R(1'-1'',t'-t'')\,G_R(3-3'',t-t'')\,H_{I31}(3'',1'') \\ \times \Psi^{(0)}(1''t'',2't',3''t'')\Big] + (2\text{ permutations}) \qquad (7.11)\end{aligned}$$

where we have used equations (3.9), (3.13), (4.10), (4.11) and (7.5). Comparison with equation (7.8) for equal times shows that they differ by factors $\frac{1}{2}$ in some of the terms. In order to get complete agreement in the perturbation expansions, we have to change the integral equation (7.3) to

$$\begin{aligned}\Psi(11,22,33) = i\alpha \int_{t}^{f} dt' \int d1'\,d2'\,G_R(1-1',t_1-t')\,G_R(2-2',t_2-t') \\ \times H_{I12}(1',2')\,\Psi(1't',2't',33) + (2\text{ permutations}) \\ - \int d1'\,d2'\,G_R(1-1',1-i)\,G_R(2-2',2-i) \\ \times \Psi(1'i,2'i,33) + (2\text{ permutations}) - 2i \int d1' \\ \times d2'\,d3'\,G_R(1-1',1-i)\,G_R(2-2',2-i) \\ \times G_R(3-3',3-i)\,\Psi_i(1',2',3') \qquad (7.12)\end{aligned}$$

which corresponds to a 'differential equation' of the form

$$\begin{aligned}M_1 M_2\Big[\Psi(11,22,33) - 2i \int d3'\,G_R(3-3',3-i)\,\Psi(11,22,3'i)\Big] \\ = 3i\alpha H_{I12}(1,2)\,\delta(1-2)\,\Psi(11,22,33) \qquad (7.13)\end{aligned}$$

The recursion relation now reads

$$\Psi^{(p)}(11,22,33) = i\int_i^f dt' \int d1'\,d2'\,G_R(1-1',t_1-t')\,G_R(2-2',t_2-t') \times H_{I12}(1',2')\,\Psi^{(p-1)}(1't',2',t',33) + (2\text{ permutations}) \quad (7.14)$$

which gives

$$M_1 M_2\,\Psi^{(p)}(11,22,33) = iH_{I12}(1,2)\,\delta(1-2)\,\Psi^{(p-1)}(11,22,33) + i\int d3'\,G_R(3-3',3-2)\,H_{I23}(2,3') M_1 \times \Psi^{(p-1)}(11,22,3'2) + i\int d3'\,G_R(3-3',3-1) \times H_{I31}(3',1)\,M_2\,\Psi^{(p-1)}(11,22,3'1), \quad (7.15)$$

$$M_1 M_2 \int d3'\,G_R(3-3',3-i)\,\Psi^{(p)}(11,22,3'i) = iH_{I12}(1,2)\,\delta(1-2)\int d3'\,G_R(3-3',3-i)\,\Psi^{(p-1)}(11,22,3'i) \quad (7.16)$$

in disagreement with (7.13). This is to be expected from the failure of the consistency conditions from (5.14) and (5.15). Consequently, it is best to start directly from an integral equation such as (7.12), without reference to differential equations.

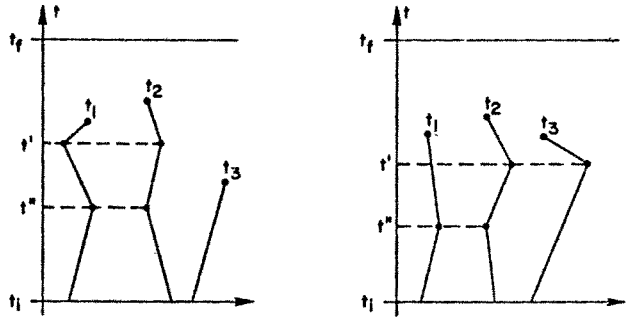

Figure 1.—Typical diagrams that represent second-order terms in the perturbation expansion of a three-particle amplitude.

The integrals from the perturbation expansion can be represented by diagrams similar to the Feynman graphs in quantum electrodynamics. Typical second-order terms are represented in Fig. 1, and we notice that, in this nonrelativistic theory, particles propagate only forward in time.

8. *Four-Particle Amplitudes*

The form of Green's theorem that corresponds to equation (7.1) is

$$\begin{aligned}\int_i^f d1\,d2\,d3\,d4 \int d\mathbf{1}\,d\mathbf{2}\,d\mathbf{3}\,d\mathbf{4}\,[&(M_1^* M_2^* \Phi) M_3 M_4 \Psi + (M_1^* M_3^* \Phi) M_2 M_4 \Psi \\ &+ (M_1^* M_4^* \Phi) M_2 M_3 \Psi + (M_2^* M_3^* \Phi) M_1 M_4 \Psi \\ &+ (M_2^* M_4^* \Phi) M_1 M_3 \Psi + (M_3^* M_4^* \Phi) M_1 M_2 \Psi \\ &- 6\Psi M_1^* M_2^* M_3^* M_4^* \Phi] \\ = \int d\mathbf{1}\,d\mathbf{2}\,d\mathbf{3}\,d\mathbf{4} \Bigg\{ &\int_i^f d1\,d2\,d3[(M_3^* \Phi) M_1 M_2 \Psi]_{4=i}^{4=f} \\ &+ (11 \text{ permutations}) + 3 \int_i^f d1\,d2[\Phi M_1 M_2 \Psi]_{3,4=i}^{3,4=f} \\ &+ (5 \text{ permutations}) + 6i \int_i^f d1[\Psi M_1^* \Phi]_{2,3,4=i}^{2,3,4=f} \\ &+ (3 \text{ permutations}) + 18[\Phi\Psi]_{1,2,3,4=i}^{1,2,3,4=f} \Bigg\} \end{aligned} \tag{8.1}$$

As in the preceding section, we can use equation (5.13) to obtain an integral equation that leads to a perturbation expansion in disagreement with the single-time theory. Alternatively, we can set

$$\begin{aligned} M_1 M_2 \Big[&\Psi(\mathbf{11}, \mathbf{22}, \mathbf{33}, \mathbf{44}) + i \int d\mathbf{3}'\, G_R(\mathbf{3} - \mathbf{3}', 3 - i)\, \Psi(\mathbf{11}, \mathbf{22}, \mathbf{3}'\, i, \mathbf{44}) \\ &+ i \int d\mathbf{4}'\, G_R(\mathbf{4} - \mathbf{4}', 4 - i)\, \Psi(\mathbf{11}, \mathbf{22}, \mathbf{33}, \mathbf{4}'\, i) - 3 \int d\mathbf{3}' \\ &\times d\mathbf{4}'\, G_R(\mathbf{3} - \mathbf{3}', 3 - i)\, G_R(\mathbf{4} - \mathbf{4}', 4 - i)\, \Psi(\mathbf{11}, \mathbf{22}, \mathbf{3}'\, i, \mathbf{4}'\, i) \Big] \\ = 6i\alpha H_{I12}(\mathbf{1}, \mathbf{2})\, &\delta(1 - 2)\, \Psi(\mathbf{11}, \mathbf{22}, \mathbf{33}, \mathbf{44}) \end{aligned} \tag{8.2}$$

in this way we get the integral equation

$$\begin{aligned} \Psi(\mathbf{11}, \mathbf{22}, \mathbf{33}, \mathbf{44}) = i\alpha \int_i^f dt' \int d\mathbf{1}'\, d\mathbf{2}'\, &G_R(\mathbf{1} - \mathbf{1}', t_1 - t')\, G_R(\mathbf{2} - \mathbf{2}', t_2 - t') \\ &\times H_{I12}(\mathbf{1}', \mathbf{2}')\, \Psi(\mathbf{1}'\, t', \mathbf{2}'\, t', \mathbf{33}, \mathbf{44}) \\ &+ (5 \text{ permutations}) + i \int d\mathbf{1}'\, d\mathbf{2}'\, d\mathbf{3}'\, G_R(\mathbf{1} - \mathbf{1}', 1 - i) \\ &\times G_R(\mathbf{2} - \mathbf{2}', 2 - i)\, G_R(\mathbf{3} - \mathbf{3}', 3 - i)\, \Psi(\mathbf{1}'\, i, \mathbf{2}'\, i, \mathbf{3}'\, i, \mathbf{44}) \\ &+ (3 \text{ permuations}) - 3 \int d\mathbf{1}'\, d\mathbf{2}'\, d\mathbf{3}'\, d\mathbf{4}' \\ G_R \times (\mathbf{1} - \mathbf{1}', 1 - i)\, &G_R(\mathbf{2} - \mathbf{2}', 2 - i)\, G_R(\mathbf{3} - \mathbf{3}', 3 - i) \\ &\times G_R(\mathbf{4} - \mathbf{4}', 4 - i)\, \Psi_i(\mathbf{1}', \mathbf{2}', \mathbf{3}', \mathbf{4}') \end{aligned} \tag{8.3}$$

The corresponding perturbation expansion starts with

$$\Psi^{(0)}(11,22,33,44) = \int d1'\,d2'\,d3'\,d4'\,G_R(1-1',1-i)\,G_R(2-2',2-i) \times G_R(3-3',3-i)\,G_R(4-4',4-i)\,\Psi_i(1',2',3',4') \tag{8.4}$$

which satisfies the initial condition. The recursion formula

$$\Psi^{(p)}(11,22,33,44) = i\int_i^f dt' \int d1'\,d2'\,G_R(1-1',t_1-t')\,G_R(2-2',t_2-t') \times H_{I12}(1',2')\,\Psi^{(p-1)}(1't',2't',33,44) + (5 \text{ permutations}) \tag{8.5}$$

gives the higher-order terms. We write out the second-order term

$$\begin{aligned}\Psi^{(2)}(11,22,33,44) = -\int_i^f dt' \int d1'\,d2'\,G_R(1-1',t_1-t') \\ \times G_R(2-2',t_2-t')\,H_{I12}(1',2')\Big[\int_i^f dt'' \int d1''\,d2'' \\ \times G_R(1'-1'',t'-t'')\,G_R(2'-2'',t'-t'') \\ \times H_{I12}(1'',2'')\,\Psi^{(0)}(1''t'',2''t'',33,44) \\ + (5 \text{ permutations}) + \int_i^f dt'' \int d1''\,d3'' \\ \times G_R(1'-1'',t'-t'')\,G_R(3-3'',t_3-t'')\,H_{I31}(3'',1'') \\ \times \Psi^{(0)}(1''t'',2't',3''t'',44) + (23 \text{ permutations}) \\ + 2\int_i^f dt'' \int d3''\,d4''\,G_R(3-3'',t_3-t'') \\ \times G_R(4-4'',t_4-t'')\,\Psi^{(0)}(1't',2't',3''t'',4''t'') \\ + (2 \text{ permutations})\Big]\end{aligned} \tag{8.6}$$

where the permutations refer to the whole terms and not just the part in the bracket, and for the last term we do not obtain a new one by interchanging primed and double-primed variables.

It is now straightforward to show that the equal-time amplitudes coincide with those obtained from the single-time formalism. The graphs that occur for the second-order contribution to the wave function are shown in Fig. 2.

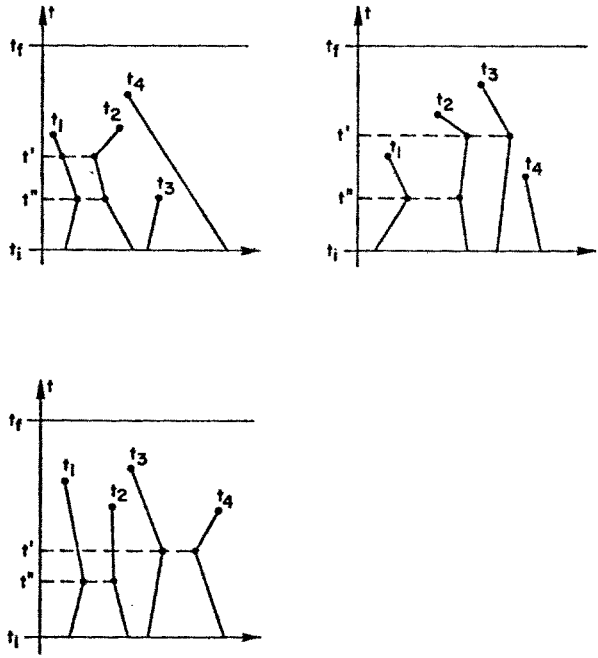

Figure 2.—Typical diagrams that represent second-order terms in the perturbation expansion of a four-particle amplitude.

9. *Bound States*

In the single-time theory of Section 4, bound states are identified with stationary states of negative energy. They can be obtained either from the time-independent Schrödinger equation or from the equivalent homogeneous integral equation.

The role of stationary states in a relativistic theory is far less clear. They are incompatible with the interpretation of the wave function in terms of probability amplitudes and the specification of initial and final conditions. Instead, bound states can be related to quasi-stationary states (Marx, 1970b; Walter & Marx, 1971), which do not correspond to solutions of the complete differential equation; the difference in the resulting energy levels is small compared to the Lamb shift (Marx, 1972b).

It is even more difficult to decide on the properties of bound states in a many-times formalism for two or more particles. Concepts such as a relative time for two particles (Bethe & Salpeter, 1951, 1957) do not lend themselves to a clear physical interpretation, and we prefer to formulate this problem in terms of an equal-time amplitude.

The differential equation (5.13) becomes meaningless for equal times, and we find it better to start from the integral equation, if possible. This approach also avoids ambiguities for more than two particles.

We therefore try a solution of the form

$$\Psi(\mathbf{1}t, \mathbf{2}t) = \psi(\mathbf{1}, \mathbf{2}) \exp(-iEt) \tag{9.1}$$

and substitute it into equation (6.5); we obtain

$$\begin{aligned}\psi(\mathbf{1}, \mathbf{2}) \exp(-iEt) = i\alpha \int_{t_i}^{t} dt' \int d\mathbf{1}' d\mathbf{2}' \, G_R(\mathbf{1} - \mathbf{1}', t - t') G_R(\mathbf{2} - \mathbf{2}', t - t') \\ \times H_{I12}(\mathbf{1}', \mathbf{2}') \psi(\mathbf{1}', \mathbf{2}') \exp(-iEt') - i \int d\mathbf{1}' \\ \times G_R(\mathbf{1} - \mathbf{1}', t - t_i) \Psi(\mathbf{1}' t_i, \mathbf{2}t) - i \int d\mathbf{2}' \\ \times G_R(\mathbf{2} - \mathbf{2}', t - t_i) \Psi(\mathbf{1}t, \mathbf{2}' i) + \int d\mathbf{1}' d\mathbf{2}' \\ \times G_R(\mathbf{1} - \mathbf{1}', t - t_i) G_R(\mathbf{2} - \mathbf{2}', t - t_i) \\ \times \psi(\mathbf{1}', \mathbf{2}') \exp(-iEt_i)\end{aligned} \tag{9.2}$$

This equation still involves the amplitude for different times, but for a stationary state we let $t_i \to -\infty$ and set the Green functions for an infinite time argument equal to zero, as shown by equation (3.8). Equation (9.2) then reduces to

$$\begin{aligned}\psi(\mathbf{1}, \mathbf{2}) = i\alpha \int_{-\infty}^{t} dt' \int d\mathbf{1}' d\mathbf{2}' \, G_R(\mathbf{1} - \mathbf{1}', t - t') G_R(\mathbf{2} - \mathbf{2}', t - t') \\ \times H_{I12}(\mathbf{1}', \mathbf{2}') \psi(\mathbf{1}', \mathbf{2}') \exp[iE(t - t')]\end{aligned} \tag{9.3}$$

The product of the Green function for equal times reduces to the two-particle Green function, and we have seen how equation (4.15) reduces to (4.14) in the single-time theory. We can also show this directly by using the substitution (4.16) and equations (1.4.21) and (2.4.31) of Erdélyi *et al.* (1954); in this way we reduce equation (9.3) to

$$\begin{aligned}\psi(\mathbf{1}, \mathbf{2}) = \alpha \int d\mathbf{1}' d\mathbf{2}' [Em^2/(2\pi^3 \xi^2)] \\ \times K_2[(-2mE\xi^2)^{1/2}] H_{I12}(\mathbf{1}', \mathbf{2}') \psi(\mathbf{1}', \mathbf{2}')\end{aligned} \tag{9.4}$$

where

$$\xi^2 = (\mathbf{x}_1 - \mathbf{x}_1')^2 + (\mathbf{x}_2 - \mathbf{x}_2')^2 \tag{9.5}$$

and we refer to equation (3.18) for the Green function.

For equal times, equation (7.12) for the three-particle amplitude becomes

$$\begin{aligned}\Psi(\mathbf{1}t, \mathbf{2}t, \mathbf{3}t) = i\alpha \int_{-\infty}^{t} dt' \int d\mathbf{1}' d\mathbf{2}' \, G_R(\mathbf{1} - \mathbf{1}', t - t') G_R(\mathbf{2} - \mathbf{2}', t - t') \\ \times H_{I12}(\mathbf{1}', \mathbf{2}') \Psi(\mathbf{1}' t', \mathbf{2}' t', \mathbf{3}t) + (2 \text{ permutations})\end{aligned} \tag{9.6}$$

which involves unequal-time amplitudes even after the boundary terms are eliminated. In order to reduce this equation to the single-time form (4.15), we would require a relationship showing that

$$\Psi(1't',2't',3t) = -i\int d3'\, G_R(3-3',t-t')\,\Psi(1't',2't',3't') \quad (9.7)$$

Since we have a stationary state and $t' < t$, we can set $t' = t_i$. We have noticed for the perturbation expansion that only the term that obeys the homogeneous equation remains if we set $t_1 = t_2 = t_i$; in this case we are led to equation (9.7). We also point out that equation (7.12) reduces to an identity for $t_1 = t_2 = t_i$.

The four-particle amplitude obeys

$$\Psi(1t,2t,3t,4t) = i\alpha \int_{-\infty}^{t} dt' \int d1'\,d2'\, G_R(1-1',t-t')\,G_R(2-2',t-t') \times H_{I12}(1',2')\,\Psi(1't',2't',3t,4t) + (5 \text{ permutations}) \quad (9.8)$$

and in this case we do not expect $\Psi(1't', 2't', 3t, 4t)$ to obey the homogeneous equation that would reduce this equation to the form of (4.15).

It is possible that a more elaborate approach to define bound states in a many-times formalism would lead to proper equations for four or more particles, but we are not interested in this problem in the relativistic case at the present time.

10. *Concluding Remarks*

We have presented an extension of the many-times formalism to the Coulomb interaction in nonrelativistic quantum mechanics. Actually, we have not used any particular properties of this interaction, and our equations apply to any equal-time two-particle interaction.

The approach that is found to reduce to the correct single-time theory is based on an integral equation; differential equations are limited by the singular nature of the interaction and by problems with integrability conditions. The perturbation expansion in the case of a scattering problem can be visualized in terms of propagation of free particles between two-particle interactions which occur at equal times. The formulation of a bound-state problem offers no difficulties for two particles, becomes less simple for three and was not solved for four or more particles.

The single-time theory is perfectly adequate for nonrelativistic motion of particles, and we have worked on this problem primarily in order to use similar equations in the relativistic generalization. In such a context, a many-times formalism coupled to causal specification of boundary conditions was found to allow for a description of pair creation and annihilation in relativistic quantum mechanics.

The equal-time feature of the interaction is also of more general interest than usually assumed in a relativistic theory. While the special role of time

in classical relativistic physics is limited in covariant formulations, a probabilistic interpretation of the corresponding quantum theory is closely related to an observer who provides a preferred time-like direction in Minkowski space (Marx, 1970c). In terms of this observer, the electromagnetic interactions of charged particles separate in a gauge-independent manner into a Coulomb interaction between particles and an interaction of the particles with the dynamical radiation field. In terms of the particle times and the field time that are the parameters of the state vectors, these interactions occur at equal times.

It is evident at different stages in our calculations how some discrepancies are introduced through the use of products of distributions, even such generally well-behaved ones as θ-functions. There obviously is a need for a framework in which such products can be consistently defined.

Further generalizations of the many-times formalism to the relativistic Coulomb problem and the interaction of particles with the radiation field would lead to a form of quantum electrodynamics in which some or all of the problems with divergences are absent, while remaining close to the spirit of a simple approach.

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